Homomorphisms of Hesitant Fuzzy Subgroups

Deepak D and Sunil Jacob John

Abstract -- Hesitant fuzzy subgroup defined on a group G generalizes the idea of fuzzy subgroups. It mainly focuses on the multiplicity of values encountered when dealing with hesitant fuzzy sets. In this paper we introduce the notions of a homomorphism and isomorphism between two hesitant fuzzy subgroups and study their effects on hesitant fuzzy subgroups. Some results regarding the homomorphisms between some quotient subgroups are also discussed.

Index Terms— Hesitant fuzzy subgroups, normal hesitant fuzzy subgroup, homomorphism and isomorphism of hesitant fuzzy subgroups.

1 Introduction

After the introduction of Fuzzy sets by L A Zadeh (1965) [9], several extensions have been developed. One of the recent extensions is the introduction of Hesitant fuzzy sets by V Torra [4, 5]. Hesitant fuzzy set (HFS) permits the membership having a set of possible values. It has applications in many decision making problems [6, 7, 8].

In 1971, A Rosenfeld [3] introduced the notion of fuzzy subgroups. Since then a lot of work has been done on fuzzy algebraic structures. In this paper we study the homomorphisms on hesitant fuzzy subgroups. In section 2 we discuss some preliminaries regarding HFS's. Section 3 studies notions of HF groups and normal HF groups which have already been introduced by us [2]. Section 4 studies the effects of homomorphisms on these structures.

2 Basic Concepts

This section introduces the basic concepts in Hesitant fuzzy set theory.

Definition 2.1 ([5]). Let X be a reference set then a Hesitant fuzzy set(HFS) on X is defined in terms of a function h that when applied to X returns a subset of [0, 1] h : X → P [0, 1] where P [0; 1] denotes power set of [0, 1].

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The empty hesitant set, the full hesitant set, the set to represent complete ignorance for x and the nonsense set are defined as follows:

Empty set : $h_0(x) = \{0\}$

Full set: $h_X(x) = \{1\}$

Complete ignorance $h(x) = [0,1]$

Set for a nonsense $x : h(x) = \phi$

Given an hesitant fuzzy set h, its lower and upper bound are defined as follows:

$h^-(x) = \min h(x)$

$h + (x) = \max h(x)$

For convenience we call $h(x)$ a hesitant fuzzy element (HFE) [7]. Let $l(h(x))$ be the number of values in $h(x)$.

Definition 2.2 ([7]). Score for a HFE, $s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma$ is called the score function of h.

Note: If the HFE is infinite then $s(h(x)) = \frac{1}{2}(\inf(h(x)) + \sup(h(x)))$..

Definition 2.3. Let $h \in HF(X)$. Then the set $\underset{x \in X}{\cup} h(x)$ is called the image of h and is denoted by $h(X)$. The set $\{x | x \in X, s(h(x)) > 0\}$, is called the support of h and is denoted by h^*. h is called finite hesitant fuzzy set if h^* is a finite set, and an infinite hesitant fuzzy set otherwise.

Definition 2.4. Let $Y \subseteq X$ and $A \subseteq [0,1]$. We define $A_Y \in HF(X)$ as follows:

$$A_Y(x)=\begin{cases} A, & \text{for } x\in Y \\ \{0\} & \text{for } x\in X\setminus Y\end{cases}$$

Definition 2.5 ([2]). Hesitant Equality:
Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is equal to h_2 (denoted $h_1=h_2$) iff $h_1(x)=h_2(x)$ and h_1 is hesitantly equal to h_2 (denoted $h_1\approx h_2$ iff $s(h_1(x))=s(h_2(x))\forall x\in X$.

Definition 2.6 ([2]). Hesitant subset:
Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is a hesitant subset of h_2 (denoted by $h_1\preceq h_2$) iff $s(h_1(x))\le s(h_2(x))\forall x\in X$.

Definition 2.7 ([5]). Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their union represented by $h_1\cup h_2$ is defined as

$$(h_1\cup h_2)(x)=\{\gamma\in(h_1(x)\cup h_2)(x)/\gamma\ge\max(h_1^-,h_2^-)\}$$
$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\max\{\gamma_1,\gamma_2\}.$$

Definition 2.8 ([5]). Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their intersection represented by $h_1\cap h_2$ is defined as

$$(h_1\cap h_2)(x)=\{\gamma\in(h_1(x)\cup h_2(x)/\gamma\le\min(h_1^+,h_2^+))\}$$
$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\min\{\gamma_1,\gamma_2\}.$$

Definition 2.9. Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, we define a score based intersection of h_1 and h_2 (denoted by $h_1\tilde{\wedge}h_2$) as

$$h_1\tilde{\wedge}h_2(x)=\begin{cases} h_1(x) & \text{if } h_1(x)\prec h_2(x)\\ h_2(x) & \text{if } h_2(x)\prec h_1(x)\\ h_1(x)\cup h_2(x) & \text{if } h_1(x)\approx h_2(x)\end{cases}$$

and a score based union of h_1 and h_2 (denoted by) $h_1\tilde{\vee}h_2$) as

$$h_1\tilde{\vee}h_2(x)=\begin{cases} h_1(x) & \text{if } h_1(x)\succ h_2(x)\\ h_2(x) & \text{if } h_2(x)\succ h_1(x)\\ h_1(x)\cup h_2(x) & \text{if } h_1(x)\approx h_2(x)\end{cases}$$

For a collection, $\{h_i|i\in I\}$ of hesitant fuzzy subsets of X, where I is a non empty index set, we have $\forall x\in X$

$$(\tilde{\vee}_{i\in I}\, h_i)(x)=\tilde{\vee}_{i\in I}\, h_i(x)$$
$$(\tilde{\wedge}_{i\in I}\, h_i)(x)=\tilde{\wedge}_{i\in I}\, h_i(x)$$

Definition 2.10. (Extension Principle). Let *f* be a function from X into Y, and let $h_1\in HF(X)$ and $h_2\in HF(Y)$. Define the hesitant fuzzy subsets $f(h_1)\in HF(Y)$ and $f^{-1}(h_2)\in HF(Y)$ by $\forall y\in Y$

$$f(h_1)(y)=\begin{cases}\tilde{\vee}\{h_1(x)|x\in X,\, f(x)=y\}, \text{if } f^{-1}(y)\ne\phi,\\ \{0\}, \text{otherwise}\end{cases}$$

and $\forall x\in x, f^{-1}(h_2)(x)=h_2(f(x))$., Then $f(h_1)$ is called the image of h_1 under *f* and $f^{-1}(h_2)$ is called the pre image of h_2 under *f*.

Theorem 2.11. Let *f* be a function from X into Y and *g* a function from Y into Z. Then the following assertions hold.

1. $h_1\preceq h_2\Rightarrow f(h_1)\preceq f(h_2)\ \forall h_1,h_2\in HF(X)$
2. $h_1\preceq h_2\Rightarrow f^{-1}(h_1)\preceq f^{-1}(h_2)\ \forall h_1,h_2\in HF(Y)$
3. $f^{-1}(f(h_1))\succeq h_1\ \forall h_1\in HF(X)$.
   In particular if *f* is a surjection, then $f^{-1}(f(h_1))=h_1\quad \forall h_1\in HF(X)$.
4. $f(f^{-1}(h_2))\preceq h_2\ \forall h_2\in HF(X)$.
   In particular if *f* is a surjection, then $f(f^{-1}(h_2))=h_2\ \forall h_2\in HF(Y)$.
5. $f(h_2)\preceq h_2\Leftrightarrow h_1\le f^{-1}(h_2)\ \forall h_1\in HF(X)$ and $h_2\in HF(Y)$.
6. $g(f(h_1))=(g\circ f)(h_1)\ \forall h_1\in HF(X)$ and $f^{-1}(g^{-1}(h_3))=(g\circ f)^{-1}(h_3)\ \forall h_3\in HF(Z)$.

3 Hesitant Fuzzy Subgroups and Normal Hesitant Fuzzy Subgroups

This section introduces the concept of composition in the case of Hesitant fuzzy sets. Hesitant fuzzy sub-group is defined and certain results regarding them are discussed. Normal Hesitant fuzzy subgroup is an important concept when it comes to the study of Hesitant fuzzy group theory. This section moves on to discuss various results regarding them.

Let *G* denote an arbitrary group with a multiplicative binary operation and identity *e*.

Definition 3.1. We define the binary operation ∘ on *HF*(*G*) and the unary operation $^{-1}$ on *HF*(*G*) as follows: $\forall h_1,h_2\in HF(G)$ and $\forall x\in G$,

$(h_1 \circ h_2)(x) = \tilde{\vee}\{h_1(x)\,\tilde{\vee}\, h_2(x) \mid y, z \in G, yz = x\}$ and
$h_1^{-1}(x) = h(x^{-1})$

We call $h_1 \circ h_2$ the product of h_1 and h_2, and h_1^{-1} the inverse of h_1.

Definition 3.2. Let $h \in HF(G)$. Then h is called hesitant fuzzy subgroup of G if

(i) $h(xy) \succeq h(x)\,\tilde{\wedge}\, h(y) \quad \forall x, y \in G$

and

(ii) $h(x^{-1}) \succeq h(x) \quad \forall x \in G$

Denote by $HFG(G)$, the set of all Hesitant Fuzzy subgroups of G. If $h \in HFG(G)$, then let $h_* = \{x \in G \mid h(x) = h(e)\}$. From (i) of the above definition we have $h(x^n) \geq h(x) \quad \forall x \in G$, where $n \in \mathrm{N}$.

Lemma 3.3. $h \in HF(G)$ is a hesitant fuzzy subgroup iff $h(xy^{-1}) \geq h(x)\,\tilde{\vee}\, h(y) \quad \forall x, y \in G$

Lemma 3.4. Let $h \in HFG(G)$. Then $\forall x \in G$,

(i) $h(e) \succeq h(x)$

(ii) $h(e) \approx h(x^{-1})$

Lemma 3.5. If $h \in HFG(G)$ and if $x, y \in G$ with $h(x) \succeq h(y)$, then $h(xy) \approx h(x)\,\tilde{\wedge}\, h(y)$.

Definition 3.6. Let G be a group. A hesitant fuzzy subgroup of a group G is called normal if $h(x) \approx h(y^{-1}xy) \; \forall x, y \in G$. Let $NHF(G)$ denote the set of all normal hesitant fuzzy subgroups of G.

Theorem 3.7. Let $h \in NHF(G)$. Then h_* and h^* are normal subgroups of G.

Definition 3.8. Let $h \in HFG(G)$ and $x \in G$. The hesitant fuzzy subset $h(e)_{\{x\}} \circ h$ is referred to as the left coset of h with respect to x and is written $x\tilde{h}$ and $h \circ h(e)_{\{x\}}$ is referred to as the right coset of h with respect to x and is written $\tilde{h}x$. Now,

$$\begin{aligned}(h(e)_x \circ h)(a) &= \tilde{\vee}\{h(e)_x(y)\,\tilde{\wedge}\, h(z) \mid y, z \in G; yz = a\} \\ &= \tilde{\vee}\begin{cases} h(e)\,\tilde{\wedge}\, h(z) \text{for } xz = a \\ \{0\}\,\tilde{\wedge}\, h(z) \text{for } xz \neq a \end{cases} \\ &= h(e)\,\tilde{\wedge}\, h(x^{-1}a) \\ &= h(x^{-1}a)\end{aligned}$$

Therefore $x\tilde{h}(a)$ and $\tilde{h}x(a) = h(ax^{-1})$.

Note: We write $\tilde{h}$ in the notation of a coset in place of the hesitant fuzzy set h so as to differentiate between the element x and the hesitant fuzzy set h. We have that if $h \in NHF(G)$ then $x\tilde{h} = \tilde{h}x$. Thus we call $x\tilde{h}$ a coset of h (dropping the notion of left or right coset).

Theorem 3.9. let $h \in NHF(G)$. Set $G/h = \{x\tilde{h} \mid x \in G\}$. Then

1. $((x\tilde{h}) \circ (y\tilde{h})) = (xy)\tilde{h} \quad \forall x, y \in G$
2. $(G/h, \circ)$ is a group

Definition 3.10. The group $(G/h, \circ)$ defined in the above theorem, where $G/h = \{x\tilde{h} \mid x \in G\}$, is called the quotient group or the factor group of G relative to the normal hesitant fuzzy subgroup h.

Theorem 3.11. Let $h \in HFG(G)$ and let N be a normal subgroup of G. Define
$(h/N) \in HF(G/N)$ as
$(h/N)(xN) = \tilde{\vee}\{h(z) \mid z \in xN\} \; \forall x \in G$ Then
$(h/N) \in HFG(G/N)$.

Definition 3.12. The hesitant fuzzy subgroup $(h/N)(xN) = \tilde{\wedge}\{h(z) \mid z \in xN\} \; \forall x \in G$ is called the quotient hesitant fuzzy subgroup or factor hesitant fuzzy subgroup of the hesitant fuzzy subgroup h of G relative to the normal group N of G.

4 Homomorphisms and Isomorphisms

Theorem 4.1. Let $h \in HFG(G)$ and H be a group. Suppose that f is a homomorphism of G into H then $f(h) \in HFG(H)$.

Proof. Let $u, v \in H$. Suppose $u \notin f(G)$ *or* $u \notin f(G)$

Then $f(h)(u)\,\tilde{\wedge}\, f(h)(u) = \{0\} \preceq f(h)(uv)$

Now assume $u \notin f(G)$. Then $u^{-1} \notin f(G)$.

Thus $f(h)(u) = \{0\} = f(h)(u^{-1})$.

Now suppose $u = f(x)$ and $v = f(y)$ for some $x, y \in G$.

Then $f(h)(uv) = \tilde{\wedge}\{h(z) \mid z \in G, f(z) = uv\}$

$\succeq \tilde{\wedge}\{h(xy) \mid x, y \in G, f(x) = u, f(y) = v\}$ (since f is a homomorphism)

$\succeq \tilde{\wedge}\{h(x)\,\tilde{\wedge}\, h(y) \mid x, y \in G, f(x) = u, f(y) = v\}$

$\approx (\tilde{\wedge}\{h(x) \mid x \in G, f(x) = u\})\,\tilde{\wedge}\,(\tilde{\wedge}\{h(y) \mid y \in G, f(y) = v\})$

$\approx (f(h))(u)\,\tilde{\wedge}\,(f(h))(v)$

Using the property of a homomorphism that $(f(z))^{-1} = f(z^{-1})$ we have

$(f(h))(u^{-1}) = \tilde{\curlywedge}\{h(z) | z \in H, f(z) = u^{-1}\}$

$= \tilde{\curlywedge}\{h(z^{-1}) | z \in H, f(z^{-1}) = u\}.$

Hence $f(h) \in HFG(G)$.

Theorem 4.2. let H be a group and $v \in HFG(H)$. Let f be a homomorphism of G into h. Then $f^{-1}(u) \in HFG(G)$.

Proof. Let $x, y \in G$ then

$f^{-1}(v)(xy) = v(f(xy)) = v(f(x)f(y)) \succeq$

$v(f(x)) \tilde{\curlywedge} v(f(y))$

$= f^{-1}(v)(x) \tilde{\curlywedge} f^{-1}(v)(y)$

Further, $f^{-1}(v)(x^{-1}) = v(f(x^{-1})) = v(f(x)^{-1}) =$

$v(f(x)) = f^{-1}(v)(x)$

Hence $f^{-1}(v) \in HFG(G)$.

Theorem 4.3. let $h \in NHF(G)$ and h be a group. Suppose that f is a surjective homomorphism of G onto H. Then $f(h) \in NHF(H)$.

Proof. We have that $f(h) \in HFG(G)$. Now let $x, y, \in H$.

Since f is a surjection, $f(a) = x$ for some a $\in$ G .

Thus $f(h)(xyx^{-1}) = \tilde{\curlyvee}\{h(b) | b \in G, f(b) = xzx^{-1}\}$

$\approx \tilde{\curlyvee}\{h(a^{-1}ba) | b \in G, f(a^{-1}ba) = y\}$

$\approx \tilde{\curlyvee}\{h(b) | aba^{-1} \in G, f(b) = y\}$ subs b by aba^{-1}

$\approx \tilde{\curlyvee}\{h(b) | b \in G, f(b) = y\}$

$= f(h)(y)$

Hence $f(h) \in NHF(H)$

Theorem 4.4. Let H be a group and $h \in NHF(G)$. If f is a homomorphism from G into H, then $f^{-1}(h) \in NHF(G)$.

Proof. We have $f^{-1}(h) \in HFG(G)$.

Now for any $x, y \in G$, we have

$f^{-1}(h)(xyz^{-1}) = h(f(xyx^{-1})) = h(f(x)f(y)f(x^{-1})) =$

$h(f(y)) = f^{-1}(h)(y)$. Hence $f^{-1}(h) \in NHF(G)$

Definition 4.5. Let $h_1, h_2 \in HFG(G)$ and $h_1 \leq h_2$. Then h_1 is called a normal hesitant fuzzy subgroup of the hesitant fuzzy subgroup h_2, denoted by $h_1 \trianglelefteq h_2$ if $h_1(xyx^{-1}) \succeq h_1(y) \tilde{\curlywedge} h_2(x)$ $\forall x, y \in G$.

Lemma 4.6. If $h_1 \in NHF(G), h_2 \in HFG(G)$ and $h_1 \preceq h_2$, then h_1 is a normal hesitant fuzzy subgroup of h_2 .

Proof. Since $h_1 \in NHF(G)$ we have $h_1(y) \approx h_1(xyz^{-1})$.

Now $h_1(xyz^{-1}) \approx h_1(y) \succeq h_1(y) \tilde{\curlywedge} h_2$ because $h_1 \preceq h_2$.

Lemma 4.7. Every hesitant fuzzy subgroup is a normal hesitant fuzzy subgroup of itself.

Proof. $h(xyx^{-1}) \succeq h(xy) \tilde{\curlywedge} h(x^{-1})$

$\succeq h(x) \tilde{\curlywedge} h(y) \tilde{\curlywedge} h(x^{-1})$

$\succeq h(x) \tilde{\curlywedge} h(y)$

Lemma 4.8. Let $h_1, h_2 \in HFG(G)$ and $h_1 \preceq h_2$. Then h_1 is a normal hesitant fuzzy subgroup of h_2 if and only if $h_1(yx) \succeq h_1(xy) \tilde{\curlywedge} h_2(y)$ $\forall x, y \in G$.

Proof. Since $h_1 \trianglelefteq h_2$ we have

$h_1(yx) = h_1(yxyy^{-1}) \succeq h_1(xy) \tilde{\curlywedge} h_2(y)$ $\forall x, y \in G$

Conversely,

$h_1(xyx^{-1}) \succeq h_1(yx^{-1}x) \tilde{\curlywedge} h_2(x) = h_1(y) \tilde{\curlywedge} h_2(x)$

Theorem 4.9. Let $h_1, h_2 \in HFG(G)$ and h_1 be a normal hesitant fuzzy subgroup of h_2 . Then $(h_1)_*$ is a normal subgroup of $(h_2)_*$ and $(h_1)^*$ is a normal subgroup of $(h_1)^*$.

Proof. If $x \in (h_1)_*$ and $y \in (h_2)_*$ then h_1 is a normal hesitant fuzzy subgroup of h_2 implies that

$h_1(y^{-1}xy) \succeq h_1(x) \tilde{\curlywedge} h_2(y) \approx h_1(e) \tilde{\curlywedge} h_2(e) \approx h_1(e).$

Hence $y^{-1}xy \in (h_1)_*$. This shows that $(h_1)_*$ is a normal subgroup of $(h_2)_*$. Similarly we can see that $h_1(y^{-1}xy) \succeq h_1(e) \succeq \{0\}$ which shows that $(h_1)^*$ is a normal subgroup of $(h_2)^*$.

Theorem 4.10. If $h_1 \in NHF(G)$ and $h_2 \in HFG(G)$ then $h_1 \tilde{\curlywedge} h_2$ is a normal hesitant fuzzy subgroup of h_2 .

Proof. We have $h_1 \tilde{\curlywedge} h_2 \in HFG(G)$ and $h_1 \tilde{\curlywedge} h_2 \preceq h_2$.

Now $(h_1 \tilde{\curlywedge} h_2)(xyx^{-1}) = h_1(xyx^{-1}) \tilde{\curlywedge} h_2(xyx^{-1})$

$\approx h_1(y) \tilde{\curlywedge} h_2(xyx^{-1})$

$\succeq h_1(y) \tilde{\curlywedge} h_2(x) \tilde{\curlywedge} h_2(y) \tilde{\curlywedge} h_2(x^{-1})$

$\approx (h_1 \tilde{\curlywedge} h_2)(y) \tilde{\curlywedge} h_2(x)$ $\forall x, y \in G$

Hence $h_1 \tilde{\curlywedge} h_2$ is a normal hesitant fuzzy subgroup of h_2.

Theorem 4.11. Let $h_1, h_2, h_3 \in HFG(G)$ be such that h_1 and h_2 are normal hesitant fuzzy subgroups of h_2. Then $h_1 \tilde{\curlywedge} h_2$ is a normal hesitant fuzzy subgroup of h_3.

Proof. We have $h_1 \tilde{\curlywedge} h_2 \in HFG(G)$ and $h_1 \tilde{\curlywedge} h_2 \preceq h_2$.

Now $h_1 \tilde{\curlywedge} h_2(xyx^{-1}) = h_1(xyx^{-1}) \tilde{\curlywedge} h_2(xyx^{-1})$
$\succeq (h_1(y) \tilde{\curlywedge} h_3(x)) \tilde{\curlywedge} (h_2(y) \tilde{\curlywedge} h_3(x))$
$\succeq (h_1 \tilde{\curlywedge} h_2)(y) \tilde{\curlywedge} h_3(x) \quad \forall x, y \in G$

Therefore $h_1 \tilde{\curlywedge} h_2$ is a normal hesitant fuzzy subgroup of h_3.

Theorem 4.12. Let $h_1, h_2 \in HFG(G)$ and h_1 be normal hesitant fuzzy subgroup of h_2. Let H be a group and f a homomorphism form G into H. Then $f(h_1)$ is a normal hesitant fuzzy subgroup of $f(h_2)$.

Proof. We have $f(h_1), f(h_2) \in HFG(H)$ and $f(h_1) \preceq f(h_2)$. Now

$(f(h_1))(xyx^{-1}) = \tilde{\curlyvee}\{h_1(z) \mid z \in G, f(z) = xyx^{-1}\}$
$\succeq \tilde{\curlyvee}\{h_1(uvu^{-1}) \mid u, v \in G, f(u) = x, f(v) = y\}$
$\succeq \tilde{\curlyvee}\{h_1(u) \tilde{\curlywedge} h_2(u) \mid u, v \in G, f(u) = x, f(v) = y\}$
$\succeq (\tilde{\curlyvee}\{h_1(v) \mid v \in G, f(v) = y\}) \tilde{\curlywedge} (\tilde{\curlywedge}\{h_2(u) \mid u \in G, f(u) = x\})$
$\approx (f(h_1))(y) \tilde{\curlywedge} (f(h_2))(x) \ \forall x, y \in H$

Hence $f(h_1)$ is a normal hesitant fuzzy subgroup of $f(h_2)$.

Theorem 4.13. Let H be a group. Let $h_1, h_2 \in HFG(H)$ and h_1 be normal hesitant fuzzy subgroups of h_2. Let f be a homomorphism from G into H. Then $f^{-1}(h_1)$ is normal hesitant fuzzy subgroup of $f^{-1}(h_2)$

Proof. We have $f^{-1}(h_1), f^{-1}(h_2) \in HFG(G)$ and $f^{-1}(h_1) \preceq f^{-1}(h_2)$. Now,

$f^{-1}(h_1)(xyx^{-1}) = h_1(f(xyx^{-1}))$
$\approx h_1(f(x) f(y) f(x^{-1}))$
$\succeq h_1(f(y)) \tilde{\curlywedge} h_2(f(x))$
$= f^{-1}(h_1)(y) \tilde{\curlywedge} f^{-1}(h_2) \quad \forall x, y \in G$

Hence $f^{-1}(h_1)$ is a normal hesitant fuzzy subgroup of $f^{-1}(h_2)$.

Definition 4.14. Let G and H be groups and let $h_1 \in HFG(G)$ and $h_2 \in HFG(H)$.

1. A homomorphism f of G onto H is called a weak homomorphism of h_1 into h_2 if $f(h_1) \preceq h_2$. If f is a weak homomorphism of h_1, then we say that h_1 is weakly homomorphic to h_2 and we write $h_1 \overset{f}{\sim} h_2$.
2. An isomorphism f of G onto H is called a weak isomorphism of h_1 into h_2 if $f(h_1) \preceq h_2$. If f is a weak isomorphism of h_1 into h_2. Then we say that h_1 is weakly isomorphic to h_2 and we write $h_1 \overset{f}{\simeq} h_2$.
3. A homomorphism f of G onto H is called a homomorphism of h_1 onto h_2, if $f(h_1) \approx h_2$. If f is a homomorphism of h_1 onto h_2, then we say that h_1 is homomorphic to h_2 and we write $h_1 \overset{f}{\approx} h_2$.
4. An isomorphism f of G onto H is called an isomorphism of h_1 onto h_2 if $f(h_1) \approx h_2$. If f is a isomorphism of h_1 onto h_2, then we say that h_1 is isomorphic to h_2 and we write $h_1 \overset{f}{\cong} h_2$.

Note: Let $h_1, h_2 \in HFG(G)$ and $h_1 \trianglelefteq h_2$. Then h_1^* is a normal subgroup of h_2^* by theorem [4.18] and $h_2 | h_2^*$ is a hesitant fuzzy subgroup of h_2^*. Then the factor hesitant fuzzy subgroup $h_2 | h_2^*$ relative to h_1^* exists.

Definition 4.15. Let $h_1, h_2 \in HFG(G)$ and $h_1 \trianglelefteq h_2$. The factor hesitant fuzzy subgroup $h_2 | h_2^*$ relative to h_1^* is called the quotient subgroup (or factor subgroup of h_2 relative to h_1) and is denoted by h_2 / h_1.

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Theorem 4.16. Let $h_1, h_2 \in HFG(G)$ and h_1 be a normal hesitant fuzzy subgroup of h_2. Then $h_2 \mid h_2^* \overset{f}{\approx} h_2 / h_1$.

Proof. Let *f* be the natural homomorphism from h_2^* onto h_2^* / h_1^*. Then $f\left(h_2 \mid h_2^*\right) =$

$$\tilde{\vee}\left\{\left(h_2 \middle| h_2^*(z)\right) \middle| z \in h_2^*, f(z) = xh_1^*\right\}$$

$$= \tilde{\vee}\left\{\left(h_2(y)\right) \middle| y \in xh_1^*\right\}$$

$$= \left(h_2/h_1\right)\left(xh_1^*\right) \quad \forall x \in h_2^*$$

$$\therefore \; h_2 \mid h_2^* \overset{f}{\approx} h_2 / h_1 .$$

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